

SYMBOLIC MODELS AND CONTROL OF DISCRETE-TIME PIECEWISE AFFINE SYSTEMS: AN APPROXIMATE SIMULATION APPROACH

GIORDANO POLA¹ AND MARIA D. DI BENEDETTO²

ABSTRACT. Symbolic models have been recently used as a sound mathematical formalism for the formal verification and control design of purely continuous and hybrid systems. In this paper we propose a sequence of symbolic models that approximates a discrete-time Piecewise Affine (PWA) system in the sense of approximate simulation and converges to the PWA system in the so-called simulation metric. Symbolic control design is then addressed with specifications expressed in terms of non-deterministic finite automata. A sequence of symbolic control strategies is derived which converges, in the sense of simulation metric, to the maximal controller solving the given specification on the PWA system.

1. INTRODUCTION

Piecewise Affine (PWA) systems have been extensively studied in the past and important research advances have been achieved, which comprise research topics on stability and stabilizability, observability, controllability, identification, optimal control and reachability. In spite of a well established literature on PWA systems, it is known that reachability problems for PWA systems are undecidable [HKPV98]. This poses serious difficulties for the formal verification and control design of such systems and spurred some researchers to approach the analysis and control of PWA systems through approximating techniques and in particular, by resorting to symbolic models. A symbolic model of a continuous or hybrid system is a finite state automaton in which a symbolic state corresponds to an aggregate of continuous states and a symbolic control label to an aggregate of continuous control inputs. Symbolic models have been employed as an effective tool to address stabilizability problems, formal verification and control design of PWA systems. Symbolic models for continuous-time PWA systems and multi-affine control systems have been studied in [HCS06] and [BH06], respectively. Discrete-time PWA systems have been considered in [MBL10, MBL12, YB10, TYB⁺10, YTC⁺12]. The work in [MBL10, MBL12] explores the use of symbolic models for stabilizability problems while the work in [YB10] for solving formal verification problems; these papers consider PWA systems with no control inputs. The work in [TYB⁺10, YTC⁺12] instead, considers PWA systems with control inputs and uses symbolic models for solving control problems with temporal logic-types specifications. While being provably correct, the results in [YB10, TYB⁺10, YTC⁺12] do not quantify the conservativeness of the approach in the formal verification and control design of PWA systems. Quantifying conservativeness is important to evaluate how far the solutions based on symbolic models are from the corresponding solutions in the pure hybrid domain. In this paper we propose a framework based on the notion of approximate simulation [GP07] in which the accuracy of the approximation schemes is formally quantified and convergence properties are derived. We construct a sequence of symbolic models that approximate a PWA system in the sense of approximate simulation, so that the distance between the symbolic models and the PWA system can be quantified through the notion of simulation metric. These symbolic models can be effectively constructed by leveraging well-known results on polytopes' operations. The sequence is proven to converge in the simulation metric to the PWA system. A fixed point in the operator generating the sequence is shown to be equivalent to the PWA system, in the sense of bisimulation. Symbolic control design of PWA systems is then addressed where specifications are expressed

The research leading to these results has been partially supported by the Center of Excellence DEWS and received funding from the European Union Seventh Framework Programme [FP7/2007-2013] under grant agreement n.257462 HYCON2 Network of excellence.

in terms of non-deterministic finite automata. We propose a sequence of symbolic control strategies that solve the symbolic control design problem with increasing accuracy. This sequence is explicitly derived from the sequence of symbolic models and can be effectively computed by using well-known results on polytopes' operations and graph theory. The sequence is proven to converge in the simulation metric to the maximal controller solving the given specification on the original PWA system.

This paper is organized as follows. In Section 2 we introduce the notation and preliminary definitions and in Section 3 the class of PWA systems considered. In Section 4 we introduce a pseudo-metric on the space of polytopes and in Section 5 we recall the notions of system and approximate simulation. In Section 6 we propose a sequence of symbolic models that are employed in Section 7 to solve symbolic control design problems. Finally Section 8 offers some concluding remarks.

2. NOTATION AND PRELIMINARY DEFINITIONS

We denote by 2^X the set of subsets of a set X . We identify a binary relation $\mathcal{R} \subseteq X \times Y$ with the map $\mathcal{R} : X \rightarrow 2^Y$ defined by $y \in \mathcal{R}(x)$ if and only if $(x, y) \in \mathcal{R}$. Given a relation $\mathcal{R} \subseteq X \times Y$, the symbol \mathcal{R}^{-1} denotes the inverse relation of \mathcal{R} , i.e. $\mathcal{R}^{-1} := \{(y, x) \in Y \times X : (x, y) \in \mathcal{R}\}$. A graph is an ordered pair $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ comprising a set \mathcal{N} of nodes together with a set $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ of edges. Given a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, two nodes $N_1, N_2 \in \mathcal{N}$ are called connected if \mathcal{G} contains a path from N_1 to N_2 . A graph is said to be connected if every pair of nodes in the graph are connected. A connected component is a maximal connected subgraph of \mathcal{G} . The symbols \mathbb{Z} , \mathbb{N} , \mathbb{R} , \mathbb{R}^+ and \mathbb{R}_0^+ denote the set of integers, non-negative integers, reals, positive and non-negative reals, respectively. The symbol $\|\cdot\|$ denotes the infinity norm. Given $i_1, i_2 \in \mathbb{N} \cup \{\infty\}$ with $i_1 < i_2$ we denote by $[i_1; i_2]$ the set $\{i_1, i_1 + 1, \dots, i_2\}$. The symbols $\text{cl}(X)$, $\text{int}(X)$ and $\text{conv}(X)$ denote respectively, the topological closure, the set of interior points, and the convex hull of a set $X \subset \mathbb{R}^n$. A polyhedron $P \subseteq \mathbb{R}^n$ is a set obtained by the intersection of a finite number of (open or closed) half-spaces. A polytope is a bounded polyhedron. The collection of vertices of a polytope P is denoted by $\mathcal{V}(P)$; note that $\mathcal{V}(P) = \mathcal{V}(\text{cl}(P)) = \mathcal{V}(\text{int}(P))$. For later use we recall the following notions.

Definition 2.1. Given a set X , a function $\mathbf{d} : X \times X \rightarrow \mathbb{R}_0^+ \cup \{\infty\}$ is a quasi-pseudo-metric for X if:

- (i) for any $x \in X$, $\mathbf{d}(x, x) = 0$.
- (ii) for any $x, y, z \in X$, $\mathbf{d}(x, y) \leq \mathbf{d}(x, z) + \mathbf{d}(z, y)$.

If condition (i) is replaced by:

- (i') $\mathbf{d}(x, y) = 0$ if and only if $x = y$,

then \mathbf{d} is said to be a quasi-metric for X . If function \mathbf{d} enjoys properties (i), (ii) and

- (iii) for any $x, y \in X$, $\mathbf{d}(x, y) = \mathbf{d}(y, x)$,

then \mathbf{d} is said a pseudo-metric for X . If function \mathbf{d} enjoys properties (i'), (ii) and (iii), it is said a metric for X . When function \mathbf{d} is a (quasi) (pseudo) metric for X , the pair (X, \mathbf{d}) is said a (quasi) (pseudo) metric space.

Definition 2.2. [RSV82] Given a quasi-pseudo-metric space (X, \mathbf{d}) , a sequence $\{x_i\}_{i \in \mathbb{N}}$ over X is left (resp. right) \mathbf{d} -convergent to $x^* \in X$, denoted $\lim_{\leftarrow} x_i = x^*$ (resp. $\lim_{\rightarrow} x_i = x^*$), if for any $\varepsilon \in \mathbb{R}^+$ there exists $N \in \mathbb{N}$ such that $\mathbf{d}(x_i, x^*) \leq \varepsilon$ (resp. $\mathbf{d}(x^*, x_i) \leq \varepsilon$) for any $i \geq N$.

3. PIECEWISE AFFINE SYSTEMS

In this paper we consider the class of discrete-time Piecewise Affine (PWA) systems described by the triplet:

$$(3.1) \quad \Sigma = (\mathbb{R}^n, \mathcal{U}, \{\Sigma_1, \Sigma_2, \dots, \Sigma_N\}),$$

where \mathbb{R}^n is the set of states, $\mathcal{U} \subseteq \mathbb{R}^m$ is the set of control inputs, and Σ_i is a constrained affine control system defined by:

$$\begin{cases} x_i(t+1) = A_i x_i(t) + B_i u_i(t) + f_i, \\ x_i(t) \in X_i, u_i(t) \in \mathcal{U}. \end{cases}$$

We suppose that the sets X_i are polyhedral, with interior, and that their collection is a partition of \mathbb{R}^n ; moreover we suppose that the set \mathcal{U} is polyhedral. We denote by $\mathbf{x}(t, x_0, \mathbf{u})$ the state reached by Σ at time $t \in \mathbb{N}$ starting from an initial state $x_0 \in \mathbb{R}^n$ with control input $\mathbf{u} : \mathbb{N} \rightarrow \mathcal{U}$. Since $\{X_i\}_{i \in [1;N]}$ is a partition of \mathbb{R}^n it is readily seen that the PWA system Σ is deterministic [LJS⁺03]. A PWA system Σ is said to be autonomous if $\mathcal{U} = \{0\}$.

4. PSEUDO-METRIC SPACE OF POLYTOPES

Consider a polytopic subset \mathcal{X} of \mathbb{R}^n and denote by $\mathcal{P}(\mathcal{X})$ the set of polytopic subsets of \mathcal{X} . We recall that the diameter $\text{Diam}(X)$ of a set $X \subseteq \mathcal{X}$ is defined by $\text{Diam}(X) = \sup_{x,y \in X} \|x - y\|$. Consider the function $\mathbf{d}_p : \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X}) \rightarrow \mathbb{R}_0^+ \cup \{\infty\}$ defined for any $X_1, X_2 \in \mathcal{P}(\mathcal{X})$ by $\mathbf{d}_p(X_1, X_2) = \text{Diam}((X_1 \setminus X_2) \cup (X_2 \setminus X_1))$.

Theorem 4.1. *$(\mathcal{P}(\mathcal{X}), \mathbf{d}_p)$ is a pseudo-metric space.*

Proof. Regarding property (i) in Definition 2.1, one obtains $\mathbf{d}_p(X, X) = \text{Diam}(\emptyset) = 0$. Property (iii) is trivially satisfied. Regarding (ii), one first notes that $X_1 \setminus X_2 \subseteq (X_1 \setminus X_3) \cup (X_3 \setminus X_2)$ for any $X_1, X_2, X_3 \in \mathcal{P}(\mathcal{X})$. Hence, the following chain of inequalities holds:

$$\begin{aligned} \mathbf{d}_p(X_1, X_2) &= \text{Diam}((X_1 \setminus X_2) \cup (X_2 \setminus X_1)) \\ &\leq \text{Diam}(((X_1 \setminus X_3) \cup (X_3 \setminus X_2)) \cup ((X_2 \setminus X_3) \cup (X_3 \setminus X_1))) \\ &\leq \text{Diam}((X_1 \setminus X_3) \cup (X_3 \setminus X_2)) + \text{Diam}((X_2 \setminus X_3) \cup (X_3 \setminus X_1)) \\ &= \mathbf{d}_p(X_1, X_3) + \mathbf{d}_p(X_3, X_2). \end{aligned}$$

□

Functions Diam and \mathbf{d}_p can be easily computed from the vertices of the polytopes involved in their arguments, as the following results show.

Proposition 4.2. $\text{Diam}(P) = \max_{v_1, v_2 \in \mathcal{V}(P)} \|v_1 - v_2\|$.

Proposition 4.3. *If $P_1 \neq P_2$ then $\mathbf{d}_p(P_1, P_2) = \max_{v_1, v_2 \in \mathcal{V}(P_1) \cup \mathcal{V}(P_2)} \|v_2 - v_1\|$.*

Proof. Since $\mathcal{V}(P_1) \cup \mathcal{V}(P_2) \subseteq \text{cl}((P_1 \setminus P_2) \cup (P_2 \setminus P_1))$, then

$$(4.1) \quad \mathbf{d}_p(P_1, P_2) \geq \max_{v_1, v_2 \in \mathcal{V}(P_1) \cup \mathcal{V}(P_2)} \|v_2 - v_1\|.$$

By Proposition 4.2 and since $\mathcal{V}(\text{conv}((P_1 \setminus P_2) \cup (P_2 \setminus P_1))) \subseteq \mathcal{V}(P_1) \cup \mathcal{V}(P_2)$, one gets:

$$(4.2) \quad \begin{aligned} \mathbf{d}_p(P_1, P_2) &= \text{Diam}((P_1 \setminus P_2) \cup (P_2 \setminus P_1)) = \text{Diam}(\text{conv}((P_1 \setminus P_2) \cup (P_2 \setminus P_1))) \\ &= \max_{v_1, v_2 \in \mathcal{V}(\text{conv}((P_1 \setminus P_2) \cup (P_2 \setminus P_1)))} \|v_2 - v_1\| \leq \max_{v_1, v_2 \in \mathcal{V}(P_1) \cup \mathcal{V}(P_2)} \|v_2 - v_1\|. \end{aligned}$$

By combining (4.1) and (4.2) the result follows. □

We conclude this section by introducing the notion of splitting policy.

Definition 4.4. Consider a finite collection of polytopes $\mathbb{P} = \{P_1, P_2, \dots, P_N\} \subset \mathcal{P}(\mathcal{X})$. A splitting policy with contraction rate $\lambda \in]0, 1[$ for \mathbb{P} is a map $\Phi_\lambda : \mathbb{P} \rightarrow 2^{\mathcal{P}(\mathcal{X})}$ enjoying the following properties:

- (i) the cardinality of $\Phi_\lambda(P_i)$ is finite.
- (ii) $\Phi_\lambda(P_i)$ is a partition of P_i .
- (iii) $\text{Diam}(P_i^j) \leq \lambda \text{Diam}(P_i)$ for all $P_i^j \in \Phi_\lambda(P_i)$.

Example 4.5. Consider a finite collection of polytopes $\mathbb{P} = \{P_1, P_2, \dots, P_N\} \subset \mathcal{P}(\mathcal{X})$. Set $\rho = \min_{i \in [1;N]} \text{Diam}(P_i)$ and denote by \mathcal{C} the collection of sets $C(i_1, \dots, i_n) = [i_1, i_1 + 1[\times \dots \times [i_n, i_n + 1[$, indexed by $i_1, \dots, i_n \in \mathbb{Z}$. Note that $\text{Diam}(C(i_1, \dots, i_n)) = 1$. Given $\lambda \in]0, 1[$ define the map $\Phi_\lambda(P_i) = \{P_i^j \in \mathcal{P}(\mathcal{X}) \mid \exists C(i_1, \dots, i_n) \in \mathcal{C} \text{ s.t. } P_i^j = P_i \cap \lambda \rho C(i_1, \dots, i_n) \neq \emptyset\}$. We now show that Φ_λ is a splitting policy with contraction rate λ . Since the sets P_i are bounded then property (i) in Definition 4.4 holds. Since \mathcal{C} is a partition of \mathbb{R}^n , then also property (ii) holds. Regarding property (iii), by definition of ρ one gets $\text{Diam}(P_i^j) \leq \text{Diam}(\lambda \rho C(i_1, \dots, i_n)) = \lambda \rho \leq \lambda \text{Diam}(P_i)$. Hence, Φ_λ is a splitting policy with contraction rate λ for \mathbb{P} .

In the sequel we denote a splitting policy with contraction rate λ by Split_λ . Given a finite collection of polytopes $\mathbb{P} = \{P_1, P_2, \dots, P_N\}$ we abuse notation by writing $\text{Split}_\lambda(\bigcup_{i \in [1;N]} P_i) = \bigcup_{i \in [1;N]} \text{Split}_\lambda(P_i)$.

5. SYMBOLIC SYSTEMS AND APPROXIMATE RELATIONS

In this paper we use the notion of systems as a unified mathematical framework to describe PWA systems as well as their symbolic models.

Definition 5.1. [Tab09] A system is a quintuple $S = (X, U, \longrightarrow, Y, H)$ consisting of a set of states X , a set of inputs U , a transition relation $\longrightarrow \subseteq X \times U \times X$, a set of outputs Y and an output function $H : X \rightarrow Y$. A transition $(x, u, x') \in \longrightarrow$ of S is denoted by $x \xrightarrow{u} x'$. A state run of S with length $T \in \mathbb{N} \cup \{\infty\}$ is a (possibly infinite) sequence of transitions $x_0 \xrightarrow{u_1} x_1 \xrightarrow{u_2} \dots \xrightarrow{u_{T-1}} x_{T-1}$ of S . An output run of S with length $T \in \mathbb{N} \cup \{\infty\}$ is a (possibly infinite) sequence of output symbols y_0, y_1, \dots, y_{T-1} such that for all y_i and y_{i+1} there exists $x_i \xrightarrow{u_{i+1}} x_{i+1}$ such that $y_i = H(x_i)$ and $y_{i+1} = H(x_{i+1})$. System S is said to be *symbolic*, if X and U are finite sets; *(pseudo) metric*, if Y is equipped with a (pseudo) metric \mathbf{d} ; *deterministic*, if for all states $x \in X$ and all inputs $u \in U$ there exists at most one transition $x \xrightarrow{u} x^+$; *non-blocking*, if for all states $x \in X$ there exists at least one transition $x \xrightarrow{u} x'$ for some $u \in U$; *with no inputs*, if U is a singleton.

For a detailed description of the notion of system and of its properties we refer to [Tab09]. In this paper we use the notions of approximate simulation and bisimulation to relate properties of PWA systems and of their symbolic systems.

Definition 5.2. [GP07] Let $S_1 = (X_1, U_1, \xrightarrow{1}, Y_1, H_1)$ and $S_2 = (X_2, U_2, \xrightarrow{2}, Y_2, H_2)$ be (pseudo) metric systems with the same output sets $Y_1 = Y_2$ and (pseudo) metric \mathbf{d} and consider a precision $\varepsilon \in \mathbb{R}_0^+$. A relation $\mathcal{R} \subseteq X_1 \times X_2$ is an ε -approximate simulation relation from S_1 to S_2 if the following conditions are satisfied:

- (i) for every $(x_1, x_2) \in \mathcal{R}$ we have $\mathbf{d}(H_1(x_1), H_2(x_2)) \leq \varepsilon$.
- (ii) for every $(x_1, x_2) \in \mathcal{R}$ existence of $x_1 \xrightarrow{u_1}_1 x'_1$ in S_1 implies existence of $x_2 \xrightarrow{u_2}_2 x'_2$ in S_2 satisfying $(x'_1, x'_2) \in \mathcal{R}$.

System S_1 is said to be ε -approximately simulated by S_2 or S_2 ε -approximately simulates S_1 , denoted $S_1 \preceq_\varepsilon S_2$, if $\mathcal{R}(X_1) = X_2$. When $\varepsilon = 0$, system S_1 is said to be exactly simulated from system S_2 , or equivalently, S_2 exactly simulates S_1 . Relation \mathcal{R} is an ε -approximate bisimulation relation between S_1 and S_2 if:

- (iii) \mathcal{R} is an ε -approximate simulation relation from S_1 to S_2 .
- (iv) \mathcal{R}^{-1} is an ε -approximate simulation relation from S_2 to S_1 .

Systems S_1 and S_2 are ε -approximately bisimilar, denoted $S_1 \cong_\varepsilon S_2$, if $\mathcal{R}(X_1) = X_2$ and $\mathcal{R}^{-1}(X_2) = X_1$. When $\varepsilon = 0$, systems S_1 and S_2 are said to be (exactly) bisimilar.

In the sequel we will work with the set $\mathcal{S}(\mathcal{P}(\mathcal{X}), \mathbf{d}_\mathbf{p})$ of pseudo-metric systems with output pseudo-metric space $(\mathcal{P}(\mathcal{X}), \mathbf{d}_\mathbf{p})$. The notions of approximate simulation and bisimulation relations induce certain metrics

on $\mathcal{S}(\mathcal{P}(\mathcal{X}), \mathbf{d}_p)$. For later purposes, we recall hereafter the simulation metric. The simulation metric from a system S_1 to a system S_2 is defined as the tightest precision ε for which S_1 is ε -approximately simulated by S_2 . More formally:

Definition 5.3. [GP07] Consider two pseudo-metric systems $S_1, S_2 \in \mathcal{S}(\mathcal{P}(\mathcal{X}), \mathbf{d}_p)$. The simulation metric $\vec{\mathbf{d}}_s$ from S_1 to S_2 is defined by $\vec{\mathbf{d}}_s(S_1, S_2) = \inf\{\varepsilon \in \mathbb{R}_0^+ \mid S_1 \preceq_\varepsilon S_2\}$.

Theorem 5.4. [GP07] *The pair $(\mathcal{S}(\mathcal{P}(\mathcal{X}), \mathbf{d}_p), \vec{\mathbf{d}}_s)$ is a quasi-pseudo-metric space¹.*

6. SEQUENCES OF SYMBOLIC MODELS

In this paper we are interested in the evolution of PWA systems within bounded subsets of the state space \mathbb{R}^n . This choice is motivated by the fact that in many applications, physical variables such as velocities, temperatures, pressures, voltages, take value within bounded sets. The expressive power of the notion of systems as in Definition 5.1 is general enough to appropriately describe the evolution of PWA systems, as formally shown hereafter.

Definition 6.1. Given the PWA system Σ and the polytopic subset \mathcal{X} of \mathbb{R}^n define the pseudo-metric system $\mathbb{S}(\Sigma) = (\mathbb{X}, \mathbb{U}, \xrightarrow{\quad}, \mathbb{Y}, \mathbb{H})$ where:

- $\mathbb{X} = \mathcal{X}$.
- $\mathbb{U} = \mathcal{U}$.
- $x \xrightarrow{u} x'$, if $x \in \mathcal{X}_i$ and $x' = A_i x + B_i u + f_i$ where $\mathcal{X}_i = X_i \cap \mathcal{X}$.
- $\mathbb{Y} = \mathcal{P}(\mathcal{X})$, equipped with the pseudo-metric \mathbf{d}_p .
- $\mathbb{H}(x) = \{x\}$.

System $\mathbb{S}(\Sigma)$ preserves important properties of the PWA system Σ when its state space \mathbb{R}^n is restricted to \mathcal{X} , as for example reachability, determinism and metric properties. (Note that $\mathbf{d}_p(\{x\}, \{y\}) = \|x - y\|$ from which, metric properties of Σ are naturally transferred to $\mathbb{S}(\Sigma)$ and vice versa.) Although system $\mathbb{S}(\Sigma)$ correctly describes the PWA system Σ within the compact set \mathcal{X} , it is not symbolic because \mathbb{X} and \mathbb{U} are not finite sets. For this reason in the sequel we introduce a sequence $\mathbb{A}_M(\Sigma)$ of symbolic models that approximate the PWA system Σ . We introduce this sequence recursively, as follows:

$$(6.1) \quad \begin{cases} \mathbb{A}_1(\Sigma) = \Omega(\mathbb{S}(\Sigma)), \\ \mathbb{A}_{M+1}(\Sigma) = \Psi(\mathbb{A}_M(\Sigma)), \end{cases}$$

where functions $\Omega : \mathcal{S}(\mathcal{P}(\mathcal{X}), \mathbf{d}_p) \rightarrow \mathcal{S}(\mathcal{P}(\mathcal{X}), \mathbf{d}_p)$ and $\Psi : \mathcal{S}(\mathcal{P}(\mathcal{X}), \mathbf{d}_p) \rightarrow \mathcal{S}(\mathcal{P}(\mathcal{X}), \mathbf{d}_p)$ are formally specified through the following Definitions 6.2 and 6.3, respectively.

Definition 6.2. Given the system $\mathbb{S}(\Sigma)$ define the system $\mathbb{A}_1(\Sigma) = (\mathbb{X}_1, \mathbb{U}_1, \xrightarrow{\quad}_1, \mathbb{Y}_1, \mathbb{H}_1)$ where:

- $\mathbb{X}_1 = \text{Split}_\lambda(\{\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_N\})$. A state in \mathbb{X}_1 is denoted by X_1^j .
- $X_1^j \xrightarrow[V^{j'}]{1} X_1^{j'}$, if the following conditions hold, where index i is such that $X_1^j \subseteq \mathcal{X}_i$:
 - there exist $x \in X_1^j$ and $u \in \mathcal{U}$ such that $A_i x + B_i u + f_i \in X_1^{j'}$.
 - $V^{j'} = \{u \in \mathcal{U} \mid \exists x \in X_1^j \text{ s.t. } A_i x + B_i u + f_i \in X_1^{j'}\}$.
- \mathbb{U}_1 is the collection of all sets $V^{j'} \subseteq \mathcal{U}$ for which $X_1^j \xrightarrow[V^{j'}]{1} X_1^{j'}$.
- $\mathbb{Y}_1 = \mathcal{P}(\mathcal{X})$, equipped with the pseudo-metric \mathbf{d}_p .
- \mathbb{H}_1 is defined as follows:
 - $\mathbb{H}_1(X_1^j) = \emptyset$, if the pairwise intersection of distinct sets $V^{j'} \in \mathbb{U}_1$ for which $X_1^j \xrightarrow[V^{j'}]{1} X_1^{j'}$, is the empty set.

¹In [GP07] quasi-pseudo-metric spaces are termed directed pseudo-metric spaces.

- $\mathbb{H}_1(X_1^j) = X_1^j$, otherwise.

Definition 6.3. Given the system $\mathbb{A}_M(\Sigma)$ define the system $\mathbb{A}_{M+1}(\Sigma) = (\mathbb{X}_{M+1}, \mathbb{U}_{M+1}, \xrightarrow{M+1}, \mathbb{Y}_{M+1}, \mathbb{H}_{M+1})$ where:

- $\mathbb{X}_{M+1} = \alpha_{M+1} \cup \beta_{M+1} \cup \gamma_{M+1}$ where:
 - $\alpha_{M+1} = \{X_M^j \in \mathbb{X}_M | \mathbb{H}_M(X_M^j) = \emptyset\}$.
 - $\beta_{M+1} = \bigcup_{X_M^j, X_M^{j'} \in \mathbb{X}_M} \text{Split}_\lambda(Z_M(j, j'))$ where:

$$Z_M(j, j') = \left\{ x \in \mathbb{H}_M(X_M^j) | \exists u \in \mathcal{U} \text{ s.t. } A_i x + B_i u + f_i \in X_M^{j'} \right\},$$

and index i is such that $X_M^j \subseteq \mathcal{X}_i$.

- $\gamma_{M+1} = \left\{ \mathcal{X} \setminus \bigcup_{X \in \alpha_{M+1} \cup \beta_{M+1}} X \right\}$.

A state in \mathbb{X}_{M+1} is denoted by X_{M+1}^j .

- $X_{M+1}^j \xrightarrow{V^{j'}}_{M+1} X_{M+1}^{j'}$, if the following conditions hold, where index i is such that $X_{M+1}^j \subseteq \mathcal{X}_i$:
 - there exist $x \in X_{M+1}^j$ and $u \in \mathcal{U}$ such that $A_i x + B_i u + f_i \in X_{M+1}^{j'}$.
 - $V^{j'} = \{u \in \mathcal{U} | \exists x \in X_{M+1}^j \text{ s.t. } A_i x + B_i u + f_i \in X_{M+1}^{j'}\}$.
- \mathbb{U}_{M+1} is the collection of all sets $V^{j'} \subseteq \mathcal{U}$ for which $X_{M+1}^j \xrightarrow{V^{j'}}_{M+1} X_{M+1}^{j'}$.
- $\mathbb{Y}_{M+1} = \mathcal{P}(\mathcal{X})$, equipped with the pseudo-metric \mathbf{d}_p .
- \mathbb{H}_{M+1} is defined as follows:
 - $\mathbb{H}_{M+1}(X_{M+1}^j) = \emptyset$, if the pairwise intersection of distinct sets $V^{j'} \in \mathbb{U}_{M+1}$ for which $X_{M+1}^j \xrightarrow{V^{j'}}_{M+1} X_{M+1}^{j'}$, is the empty set.
 - $\mathbb{H}_{M+1}(X_{M+1}^j) = X_{M+1}^j$, otherwise.

Since by definition of splitting policies, the sets in $\text{Split}_\lambda(\{\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_N\})$ and β_{M+1} are finite, system $\mathbb{A}_M(\Sigma)$ is symbolic. Note that by definition of γ_M , the collection of sets in \mathbb{X}_M is a covering of \mathcal{X} . Moreover \mathbb{X}_M becomes a partition of \mathcal{X} when the PWA system Σ is autonomous. Symbolic system $\mathbb{A}_{M+1}(\Sigma)$ can be viewed as a refinement of $\mathbb{A}_M(\Sigma)$ [CGP99]. It is readily seen that the effective construction of $\mathbb{A}_M(\Sigma)$ relies upon basic operations on polytopes.

A common issue when constructing symbolic systems of PWA systems, is the presence of the so-called *spurious transitions*, that are transitions in the symbolic systems which cannot be matched by any state evolution in the PWA systems. Whenever a pair of transitions $X_M^j \xrightarrow{V^{j_1}}_M X_M^{j_1}$ and $X_M^j \xrightarrow{V^{j_2}}_M X_M^{j_2}$ exists in $\mathbb{A}_M(\Sigma)$ such that $V^{j_1} \cap V^{j_2} \neq \emptyset$ and $X_M^{j_1} \neq X_M^{j_2}$ then, those transitions are said spurious (and state X_M^j is said to produce spurious transitions) because they cannot be mimicked by any state evolution of the (deterministic) PWA system Σ . The output function \mathbb{H}_M of the symbolic system $\mathbb{A}_M(\Sigma)$ distinguishes states X_M^j that produce spurious transitions, for which $\mathbb{H}_M(X_M^j) \neq \emptyset$, from states X_M^j that do not, for which $\mathbb{H}_M(X_M^j) = \emptyset$. From the above discussion, one would argue that when there are no spurious transitions in the symbolic system, such a symbolic system correctly describes the original PWA system Σ . The following result makes this statement formal.

Theorem 6.4. *Systems $\mathbb{S}(\Sigma)$ and $\mathbb{A}_M(\Sigma)$ are bisimilar if*

$$(6.2) \quad \mathbb{H}_M(X_M^j) = \emptyset, \forall X_M^j \in \mathbb{X}_M.$$

Proof. Define the relation $\mathcal{R} \subseteq \mathbb{X} \times \mathbb{X}_M$ such that $(x, X_M^j) \in \mathcal{R}$ if and only if $x \in X_M^j$. Consider any $(x, X_M^j) \in \mathcal{R}$. From (6.2), one gets:

$$(6.3) \quad \mathbf{d}_p(\mathbb{H}(x), \mathbb{H}_M(X_M^j)) = \mathbf{d}_p(\{x\}, \emptyset) = \text{Diam}(\{x\}) = 0,$$

from which, condition (i) in Definition 5.2 holds. We now show that also condition (ii) in Definition 5.2 is satisfied. Consider any transition $x \xrightarrow{u} x'$ in $\mathbb{S}(\Sigma)$. Since \mathbb{X}_M is a covering of \mathcal{X} there exists a state $X_M^{j'} \in \mathbb{X}_M$ such that $x' \in X_M^{j'}$ or equivalently, $(x', X_M^{j'}) \in \mathcal{R}$. By definition of $\mathbb{A}_M(\Sigma)$ there exists a transition $X_M^j \xrightarrow[V^{j'}]{M} X_M^{j'}$ in $\mathbb{A}_M(\Sigma)$, for some $V^{j'} \in \mathbb{U}_M$ with $u \in V^{j'}$. Hence, condition (ii) in Definition 5.2 holds and \mathcal{R} is an exact simulation relation from $\mathbb{S}(\Sigma)$ to $\mathbb{A}_M(\Sigma)$. Since any state $X_M^j \in \mathbb{X}_M$ is non-empty then $\mathcal{R}(\mathbb{X}) = \mathbb{X}_M$, which implies that condition (iii) is satisfied with $\varepsilon = 0$. We now show that condition (iv) holds with $\varepsilon = 0$. Consider any $(X_M^j, x) \in \mathcal{R}^{-1}$. Since \mathbf{d}_p is a pseudo-metric then $\mathbf{d}_p(\mathbb{H}_M(X_M^j), \mathbb{H}(x)) = \mathbf{d}_p(\mathbb{H}(x), \mathbb{H}_M(X_M^j))$ which by (6.3) implies condition (i) in Definition 5.2. Regarding condition (ii), consider any transition $X_M^j \xrightarrow[V^{j'}]{M} X_M^{j'}$ in $\mathbb{A}_M(\Sigma)$. Pick any $u \in V^{j'}$ and consider the transition $x \xrightarrow{u} x'$ in $\mathbb{S}(\Sigma)$. Since $\mathbb{S}(\Sigma) \preceq_0 \mathbb{A}_M(\Sigma)$ with simulation relation \mathcal{R} , there exists a transition $X_M^j \xrightarrow[V^{j''}]{M} X_M^{j''}$ in $\mathbb{A}_M(\Sigma)$ such that $(x', X_M^{j''}) \in \mathcal{R}$, or equivalently $(X_M^{j''}, x') \in \mathcal{R}^{-1}$. Since $u \in V^{j'}$ and $\mathbb{H}_M(X_M^j) = \emptyset$, then $V^{j''} = V^{j'}$ and $X_M^{j''} = X_M^{j'}$. Hence, condition (ii) in Definition 5.2 is true from which, condition (vi) holds with $\varepsilon = 0$. Therefore, \mathcal{R} is an exact bisimulation relation between $\mathbb{S}(\Sigma)$ and $\mathbb{A}_M(\Sigma)$. Finally, since \mathbb{X}_M is a covering of \mathcal{X} then $\mathcal{R}^{-1}(\mathbb{X}_M) = \mathbb{X}$, which concludes the proof. \square

Whenever a state $X_M^j \in \mathbb{X}_M$ not producing spurious transitions is found from algorithm (6.1), it is no further splitted in the higher order symbolic systems. This fact implies that the union of those sets, i.e.

$$(6.4) \quad \bigcup_{X_M^j \in \{X \in \mathbb{X}_M \mid \mathbb{H}_M(X) = \emptyset\}} X_M^j,$$

is non-decreasing (in the sense of the partial order² induced by \subseteq on $\mathcal{P}(\mathcal{X})$) with respect to the order M of the symbolic system. If a step M in the algorithm in (6.1) exists for which the set in (6.4) covers the whole set \mathcal{X} , the symbolic system $\mathbb{A}_M(\Sigma)$ becomes an exact bisimulation of the PWA system.

Corollary 6.5. *If $\mathbb{A}_{M+1}(\Sigma) = \mathbb{A}_M(\Sigma)$ then $\mathbb{A}_M(\Sigma)$ and $\mathbb{S}(\Sigma)$ are exactly bisimilar.*

Proof. If $\mathbb{A}_{M+1}(\Sigma) = \mathbb{A}_M(\Sigma)$ then $\mathbb{X}_{M+1} = \mathbb{X}_M$ which implies by Definition 6.3 that $\mathbb{H}_M(X_M^j) = \emptyset$ for all $X_M^j \in \mathbb{X}_M$. Hence, the result follows as a direct application of Theorem 6.4. \square

If conditions in the above result are satisfied we say that the algorithm in (6.1) converges in a finite number of steps. In general, the algorithm in (6.1) is not guaranteed to converge in a finite number of steps. This happens whenever there are states in the symbolic system which produce spurious transitions. We now introduce a measure of these states in the symbolic system $\mathbb{A}_M(\Sigma)$. Define $\text{Gran}(\mathbb{A}_M(\Sigma)) = \max_{X_M^j \in \mathbb{X}_M} \text{Diam}(\mathbb{H}_M(X_M^j))$ and

$$(6.5) \quad \text{Gran}(\mathbb{S}(\Sigma)) = \max_{i \in [1; N]} \text{Diam}(\mathcal{X}_i).$$

Note that since sets \mathcal{X}_i are bounded then $\text{Gran}(\mathbb{A}_M(\Sigma))$ and $\text{Gran}(\mathbb{S}(\Sigma))$ cannot be infinite. Function Gran provides a measure of the "granularity" of the symbolic system (i.e. how finer is the covering of the set \mathcal{X}), in the regard of (the only) states producing spurious transitions. The following result provides an upper bound to the distance between the PWA system Σ and the proposed symbolic systems.

Theorem 6.6. $\vec{\mathbf{d}}_s(\mathbb{S}(\Sigma), \mathbb{A}_M(\Sigma)) \leq \text{Gran}(\mathbb{A}_M(\Sigma)).$

Proof. Define the relation $\mathcal{R} \subseteq \mathbb{X} \times \mathbb{X}_M$ such that $(x, X_M^j) \in \mathcal{R}$ if and only if $x \in X_M^j$. Consider any $(x, X_M^j) \in \mathcal{R}$. If $\mathbb{H}_M(X_M^j) = \emptyset$ then $\mathbf{d}_p(\mathbb{H}(x), \mathbb{H}_M(X_M^j)) = \mathbf{d}_p(\{x\}, \emptyset) = 0 \leq \text{Gran}(\mathbb{A}_M(\Sigma))$. If $\mathbb{H}_M(X_M^j) \neq \emptyset$

²A partial order \prec over a set X is a binary relation $\prec \subseteq X \times X$ that is reflexive ($x \prec x$), antisymmetric ($x \prec y$ and $y \prec x$ imply $x = y$) and transitive ($x \prec y$ and $y \prec z$ imply $x \prec z$).

then, by definition of $\text{Gran}(\mathbb{A}_M(\Sigma))$ one gets $\mathbf{d}_p(\mathbb{H}(x), \mathbb{H}_M(X_M^j)) = \text{Diam}(\mathbb{H}_M(X_M^j)) \leq \text{Gran}(\mathbb{A}_M(\Sigma))$. Hence, condition (i) in Definition 5.2 is satisfied. Conditions (ii) and $\mathcal{R}(\mathbb{X}) = \mathbb{X}_M$ can be proven along the lines of the first part of the proof of Theorem 6.4 from which, $\mathbb{S}(\Sigma) \preceq_\varepsilon \mathbb{A}_M(\Sigma)$ with $\varepsilon = \text{Gran}(\mathbb{A}_M(\Sigma))$. Finally, the result follows from the definition of the quasi-pseudo-metric $\vec{\mathbf{d}}_s$. \square

The above result is important because it quantifies the accuracy of the proposed symbolic systems. The rest of this section is devoted to study the convergence of the symbolic system $\mathbb{A}_M(\Sigma)$ to $\mathbb{S}(\Sigma)$. We start by presenting the following technical result.

Lemma 6.7. $\text{Gran}(\mathbb{A}_{M+1}(\Sigma)) \leq \lambda \text{Gran}(\mathbb{A}_M(\Sigma))$ and $\text{Gran}(\mathbb{A}_1(\Sigma)) \leq \lambda \text{Gran}(\mathbb{S}(\Sigma))$.

Proof. By Definition 6.3, for all states X_{M+1}^j in $\mathbb{A}_{M+1}(\Sigma)$ there exist a state $X_M^{j'} \in \mathbb{X}_M$ with $\mathbb{H}_M(X_M^{j'}) \neq \emptyset$ such that $X_{M+1}^j \subseteq \text{Split}_\lambda(Z)$, for some set $Z \subseteq X_M^{j'}$. By the above condition and the definition of the splitting policy Split_λ , the inequality $\text{Diam}(\mathbb{H}_{M+1}(X_{M+1}^j)) \leq \lambda \text{Diam}(Z) \leq \lambda \text{Diam}(\mathbb{H}_M(X_M^{j'}))$ holds. Hence, by applying the maximum operator to both sides of the above inequality, one gets:

$$\begin{aligned} \text{Gran}(\mathbb{A}_{M+1}(\Sigma)) &= \max_{X_{M+1}^j \in \mathbb{X}_{M+1}} \text{Diam}(\mathbb{H}_{M+1}(X_{M+1}^j)) \\ &\leq \lambda \max_{X_M^{j'} \in \mathbb{X}_M} \text{Diam}(\mathbb{H}_M(X_M^{j'})) = \lambda \text{Gran}(\mathbb{A}_M(\Sigma)), \end{aligned}$$

which concludes the first part of the proof. The second part of the proof can be shown by using similar arguments. \square

We now have all the ingredients to present one of the main results of this paper.

Theorem 6.8. *The sequence of symbolic systems $\{\mathbb{A}_M(\Sigma)\}_{M \in \mathbb{N}}$ is left $\vec{\mathbf{d}}_s$ -convergent to the system $\mathbb{S}(\Sigma)$, i.e. $\mathbb{S}(\Sigma) = \lim_{\leftarrow} \mathbb{A}_M(\Sigma)$.*

Proof. Pick any $\varepsilon \in \mathbb{R}^+$ and choose $M_\varepsilon \in \mathbb{N}$ such that:

$$(6.6) \quad \lambda^{M_\varepsilon} \text{Gran}(\mathbb{S}(\Sigma)) \leq \varepsilon.$$

By combining Theorem 6.6 and Lemma 6.7 for all $M \geq M_\varepsilon$, we obtain:

$$\vec{\mathbf{d}}_s(\mathbb{S}(\Sigma), \mathbb{A}_M(\Sigma)) \leq \text{Gran}(\mathbb{A}_M(\Sigma)) \leq \lambda \text{Gran}(\mathbb{A}_{M-1}(\Sigma)) \leq \dots \leq \lambda^M \text{Gran}(\mathbb{S}(\Sigma)) \leq \lambda^{M_\varepsilon} \text{Gran}(\mathbb{S}(\Sigma))$$

which, combined with (6.6), concludes the proof. \square

7. SYMBOLIC CONTROL DESIGN

In this section we address symbolic control design of PWA systems where specifications are expressed in terms of non-deterministic finite automata. We start by introducing the class of control strategies considered in this paper. A control strategy is specified by a partition $\mathbf{X} = \{\mathbf{X}_i\}_{i \in I}$ of \mathcal{X} and a function:

$$(7.1) \quad \mathcal{K} : \mathbf{X} \rightarrow 2^{\mathcal{U}}.$$

Note that we are not supposing that \mathbf{X} is either finite or countable. When \mathbf{X} is a finite set, the control strategy is said symbolic. Function \mathcal{K} associates to an aggregate of states $\mathbf{X}_i \in \mathbf{X}$ an aggregate of inputs $\mathcal{K}(\mathbf{X}_i) \subseteq \mathcal{U}$ representing the collection of admissible inputs. This class of control strategies is general enough to enforce automata theory-types specifications, as shown in the sequel. Given a control strategy \mathcal{K} , we denote by $\Sigma^\mathcal{K}$ the closed-loop PWA system Σ , i.e. the system Σ in (3.1) where $u = \kappa(x) \in \mathcal{K}(\mathbf{X}_i)$ if $x \in \mathbf{X}_i$. With abuse of notation, we denote by $\mathbf{x}(t, x_0, \kappa)$ the state reached by Σ at time t starting from an initial state $x_0 \in \mathcal{X}$ with feedback control law $\kappa(x) \in \mathcal{K}(\mathbf{X}_i)$, $x \in \mathbf{X}_i$; moreover we write $\kappa \in \mathcal{K}$ when $\kappa(x) \in \mathcal{K}(\mathbf{X}_i)$ for all $x \in \mathbf{X}_i$ and $\mathcal{K} \subseteq \mathcal{K}'$ when $\mathcal{K}(\{x\}) \subseteq \mathcal{K}'(\{x\})$ for all $x \in \mathcal{X}$. We can now formally state the control design problem

considered in this paper. Consider a specification described by the following pseudo-metric symbolic system with no inputs:

$$(7.2) \quad Q = (\mathbb{X}^q, \mathbb{U}_q, \xrightarrow[q]{}, \mathbb{Y}_q, \mathbb{H}_q),$$

where:

- $\mathbb{X}^q = \{\mathcal{X}_1^q, \mathcal{X}_2^q, \dots, \mathcal{X}_{N^q}^q\}$ is a finite collection of polytopic subsets of \mathcal{X} with empty pairwise intersection. Set $\mathcal{X}^q = \bigcup_{i \in [1; N^q]} \mathcal{X}_i^q$.
- $\mathbb{U}_q = \{0\}$.
- $\xrightarrow[q]{} \subseteq \mathbb{X}^q \times \mathbb{U}^q \times \mathbb{X}^q$.
- $\mathbb{Y}_q = 2^{\mathcal{X}}$, equipped with the pseudo-metric \mathbf{d}_p .
- $\mathbb{H}_q(\mathcal{X}_i^q) = \mathcal{X}_i^q$.

We suppose that Q is non-blocking and that the collection \mathbb{X}^q of sets \mathcal{X}_i^q is contained in the partition $\{\mathcal{X}_i\}_{i \in [1; N]}$ of \mathcal{X} . The last assumption can be given without loss of generality by appropriately duplicating the dynamics of the PWA system Σ . For easy of notation we denote in the sequel a transition $\mathcal{X}_{i_1}^q \xrightarrow[q]{0} \mathcal{X}_{i_2}^q$ by $\mathcal{X}_{i_1}^q \xrightarrow[q]{} \mathcal{X}_{i_2}^q$.

Definition 7.1. A control strategy $\mathcal{K} : \mathbf{X} \rightarrow 2^{\mathcal{U}}$ is said to enforce the specification Q in (7.2) on the PWA system Σ if for all initial states $x_0 \in \mathbf{X}_i$ of Σ for which $\mathcal{K}(\mathbf{X}_i) \neq \emptyset$ and for all $\kappa \in \mathcal{K}$ there exists a (possibly infinite) state run $\mathcal{X}_{i_1}^q \xrightarrow[q]{} \mathcal{X}_{i_2}^q \xrightarrow[q]{} \dots \xrightarrow[q]{} \mathcal{X}_{i_T}^q$ of Q with length $T \in \mathbb{N} \cup \{\infty\}$ such that $\mathbf{x}(t, x_0, \kappa) \in \mathcal{X}_{i_t}^q$ and $\mathbf{x}(t+1, x_0, \kappa) \in \mathcal{X}_{i_{t+1}}^q$ for all $t \in [0; T-1]$.

Denote by $\mathbf{K}(\Sigma, Q)$ the collection of all control strategies enforcing the specification Q on Σ .

Definition 7.2. The *maximal control strategy* enforcing the specification Q on the PWA system Σ , is a control strategy $\mathcal{K}^* \in \mathbf{K}(\Sigma, Q)$ such that $\mathcal{K}(\{x\}) \subseteq \mathcal{K}^*(\{x\})$ for all $\mathcal{K} \in \mathbf{K}(\Sigma, Q)$ and all $x \in \mathcal{X}$.

Proposition 7.3. $\mathcal{K}^*(\{x\}) = \bigcup_{\mathcal{K} \in \mathbf{K}(\Sigma, Q)} \mathcal{K}(\{x\})$.

From the above result the control strategy \mathcal{K}^* exists and is unique. Note that \mathcal{K}^* is not symbolic in general. Moreover, the explicit expression of \mathcal{K}^* cannot be easily derived. For this reason in the sequel we propose a sequence of control strategies \mathcal{K}_M , approximating \mathcal{K}^* , that can be effectively computed on the basis of the symbolic systems $\mathbb{A}_M(\Sigma)$.

Definition 7.4. Consider the symbolic system $\mathbb{A}_M(\Sigma)$ in Definition 6.3 and for all $X_M^j \in \mathbb{X}_M$ define the graph $\mathcal{G}(X_M^j) = (\mathcal{N}, \mathcal{E})$ where:

- \mathcal{N} is the collection of sets $V^{j'} \in \mathbb{U}_M$ such that $X_M^j \xrightarrow[M]{V^{j'}} X_M^{j'}$.
- \mathcal{E} is the collection of all pairs $(V^j, V^{j'}) \in \mathcal{N} \times \mathcal{N}$ such that $V^j \cap V^{j'} \neq \emptyset$.

For all connected component $\mathbf{G}_i(X_M^j)$ of $\mathcal{G}(X_M^j)$ define the following sets:

- $\mathbf{U}_i(X_M^j)$ is the union of nodes of $\mathbf{G}_i(X_M^j)$.
- $\mathbf{X}_i(X_M^j)$ is the union of sets $X_M^{j'}$ for which $X_M^j \xrightarrow[M]{V^{j'}} X_M^{j'}$ and $V^{j'}$ is a node of $\mathbf{G}_i(X_M^j)$.

Define the control strategy $\mathcal{K}_M : \mathbb{X}_M \rightarrow 2^{\mathcal{U}}$ such that:

- for all $X_M^j \not\subseteq \mathcal{X}^q$, $\mathcal{K}_M(X_M^j) = \emptyset$;
- for all $X_M^j \subseteq \mathcal{X}^q$, $\mathcal{K}_M(X_M^j) = \bigcup_i \mathbf{U}_i(X_M^j)$ s.t. $\mathbf{X}_i(X_M^j) \subseteq \text{Post}_q(X_M^j)$, where $\text{Post}_q(X_M^j)$ is the union of sets $\mathcal{X}_{j'}^q \in \mathbb{X}^q$ such that $X_M^j \subseteq \mathcal{X}_j^q \in \mathbb{X}^q$ and $\mathcal{X}_j^q \xrightarrow[q]{} \mathcal{X}_{j'}^q$.

Definition 7.9. Given the symbolic control strategy \mathcal{K}_M define the pseudo-metric symbolic system $\mathbb{S}(\mathcal{K}_M) = (\mathbb{X}_M, \mathbb{U}_M, \xrightarrow{\mathcal{K}_M}, \mathbb{Y}_M, \mathbb{H}_M)$, where entities $\mathbb{X}_M, \mathbb{U}_M, \mathbb{Y}_M$ and \mathbb{H}_M are defined in Definition 6.3 and $X_M^j \xrightarrow{\mathcal{K}_M} X_M^{j'}$ if and only if $X_M^j \xrightarrow{V^{j'}} X_M^{j'}$ in $\mathbb{A}_M(\Sigma)$ and $V^{j'} \subseteq \mathcal{K}_M(X_M^j)$.

We can now give the following result that quantifies the distance between \mathcal{K}_M and \mathcal{K}^* .

Theorem 7.10. $\vec{d}_s(\mathbb{S}(\mathcal{K}_M), \mathbb{S}(\mathcal{K}^*)) \leq \text{Gran}(\mathbb{A}_M(\Sigma))$.

Proof. Define the relation $\mathcal{R} \subseteq \mathbb{X}_M \times \mathbb{X}$ such that $(X_M^j, x) \in \mathcal{R}$ if and only if $x \in X_M^j$. Consider any $(X_M^j, x) \in \mathcal{R}$. By definition of $\text{Gran}(\mathbb{A}_M(\Sigma))$ one gets $\mathbf{d}_p(\mathbb{H}_M(X_M^j), \mathbb{H}(x)) = \text{Diam}(\mathbb{H}_M(X_M^j)) \leq \text{Gran}(\mathbb{A}_M(\Sigma))$ from which, condition (i) in Definition 5.2 holds. We now show that also condition (ii) in Definition 5.2 is satisfied. Consider any transition $X_M^j \xrightarrow{\mathcal{K}_M} X_M^{j'}$ in $\mathbb{S}(\mathcal{K}_M)$. By definition of \mathcal{K}^* , $\mathcal{K}_M(X_M^j) \subseteq \mathcal{K}^*(X_M^j)$. Hence, for all $u \in \mathcal{K}_M(X_M^j)$, $x \xrightarrow{\mathcal{K}^*} x^+$. In particular, by definition of \mathcal{K}_M there exists $u \in \mathcal{K}_M(X_M^j) \subseteq \mathcal{K}^*(X_M^j)$ such that $x \xrightarrow{\mathcal{K}^*} x^+$ in $\mathbb{S}(\mathcal{K}^*)$ and $x^+ \in X_M^{j'}$ from which, condition (ii) in Definition 5.2 is satisfied. Since \mathbb{X}_M is a partition of \mathcal{X} then condition (iii) in Definition 5.2 holds. Finally, the result follows from the definition of the quasi-pseudo-metric \vec{d}_s . \square

We are now ready to present the second main result of this paper.

Theorem 7.11. *The sequence of symbolic controllers $\{\mathbb{S}(\mathcal{K}_M)\}_{M \in \mathbb{N}}$ is right \vec{d}_s -convergent to $\mathbb{S}(\mathcal{K}^*)$, i.e. $\lim_{\rightarrow} \mathbb{S}(\mathcal{K}_M) = \mathbb{S}(\mathcal{K}^*)$.*

The proof of the above result can be obtained by combining Lemma 6.7 and Theorem 7.10, along the lines of the proof of Theorem 6.8, and is therefore omitted.

8. CONCLUSION

In this paper we proposed a sequence of symbolic models that converges to a PWA system in the simulation metric. A sequence of symbolic control strategies is then derived which converges in the simulation metric to the maximal controller enforcing finite automata-types specifications on the PWA system. An application of the proposed results to practical case studies requires subsequent developments of efficient computational tools. Useful insights in this direction are reported in [YTC⁺12, PBD12]. We plan to investigate this issue in future work.

REFERENCES

- [BH06] C. Belta and L.C.G.J.M. Habets. Controlling a class of nonlinear systems on rectangles. *IEEE Transactions of Automatic Control*, 51(11):1749–1759, 2006.
- [CGP99] E.M. Clarke, O. Grumberg, and D. Peled. *Model Checking*. MIT Press, 1999.
- [GP07] A. Girard and G.J. Pappas. Approximation metrics for discrete and continuous systems. *IEEE Transactions on Automatic Control*, 52(5):782–798, 2007.
- [HCS06] L.C.G.J.M. Habets, P.J. Collins, and J.H. Van Schuppen. Reachability and control synthesis for piecewise-affine hybrid systems on simplices. *IEEE Transaction on Automatic Control*, 51(6):938–948, 2006.
- [HKPV98] T.A. Henzinger, P.W. Kopke, A. Puri, and P. Varaiya. What's decidable about hybrid automata? *Journal of Computer and System Sciences*, 57:94–124, 1998.
- [LJS⁺03] J. Lygeros, K.H. Johansson, S.N. Simic, Jun Zhang, and S.S. Sastry. Dynamical properties of hybrid automata. *IEEE Transactions on Automatic Control*, 48(1):2–17, 2003.
- [MBL10] S. Mirzazad-Barijough and J.-W. Lee. On stability and performance analysis of discrete-time piecewise affine systems. In *49th IEEE Conference on Decision and Control*, pages 4244–4249, Atlanta, USA, December 2010.
- [MBL12] S. Mirzazad-Barijough and Ji-Woong Lee. Stability and transient performance of discrete-time piecewise affine systems. *IEEE Transactions of Automatic Control*, 2012. In Press.

- [PBD12] G. Pola, A. Borri, and M.D. Di Benedetto. Integrated design of symbolic controllers for nonlinear systems. *IEEE Transactions of Automatic Control*, 57(2):534–539, February 2012.
- [RSV82] I.L. Reilly, P.V. Subrahmanyam, and M.K. Vamanamurthy. Cauchy sequences in quasi-pseudo-metric spaces. *Monatshefte für Mathematik*, 93(2):127–140, 1982.
- [Tab09] P. Tabuada. *Verification and Control of Hybrid Systems: A Symbolic Approach*. Springer, 2009.
- [TYB⁺10] J. Tumova, B. Yordanov, C. Belta, I. Cerna, and J. Barnat. A symbolic approach to controlling piecewise affine systems. In *49th IEEE Conference on Decision and Control*, pages 4230–4235, Atlanta, USA, December 2010.
- [YB10] B. Yordanov and C. Belta. Formal analysis of discrete-time piecewise affine systems. *IEEE Transactions of Automatic Control*, 55(12):2834–2840, 2010.
- [YTC⁺12] B. Yordanov, J. Tumova, I. Cerna, J. Barnat, and C. Belta. Temporal logic control of discrete-time piecewise affine systems. *IEEE Transactions of Automatic Control*, 2012. In Press.

¹ DEPARTMENT OF ELECTRICAL AND INFORMATION ENGINEERING, CENTER OF EXCELLENCE DEWS, UNIVERSITY OF L’AQUILA, VIA G. GRONCHI, 67100 L’AQUILA, ITALY

E-mail address: {giordano.pola}@univaq.it

² DEPARTMENT OF ELECTRICAL AND INFORMATION ENGINEERING, CENTER OF EXCELLENCE DEWS, UNIVERSITY OF L’AQUILA, VIA G. GRONCHI, 67100 L’AQUILA, ITALY

E-mail address: {mariadomenica.dibenedetto}@univaq.it